Intermediate Mathematical Challenge

Organised by the United Kingdom Mathematics Trust

supported by Institute and Faculty of Actuaries

Solutions and investigations

7 February 2019

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here are longer. In some cases we give alternative solutions, and we have included some exercises for further investigation.

We welcome comments on these solutions. Please send them to enquiry@ukmt.org.uk.

The Intermediate Mathematical Challenge (IMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the IMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. So for each question we have included a complete solution which does not use the fact that one of the given alternatives is correct. Thus we have aimed to give full solutions with all steps explained (or, occasionally, left as an exercise). We therefore hope that these solutions can be used as a model for the type of written solution that is expected when a complete solution to a mathematical problem is required (for example, in the Intermediate Mathematical Olympiad and similar competitions).

Enquiries about the Intermediate Mathematical Challenge should be sent to:

IMC, UKMT, School of Mathematics Satellite, University of Leeds,
Leeds LS2 9JT

☎ 0113 343 2339 enquiry@ukmt.org.uk www.ukmt.org.uk

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
B	E	D	B	D	B	B	E	C	B	C	C	A	C	A	A	D	B	E	B	D	E	A	C	D

1. What is the value of 2019 tenths?

A 2019 B 201.9 C 20.19 D 2.019 E 0.2019

SOLUTION **B**

The value of 2019 tenths is $2019 \times \frac{1}{10} = 201.9$.

FOR INVESTIGATION

1.1 Write the following as decimals.

(a) 2019 hundredths,

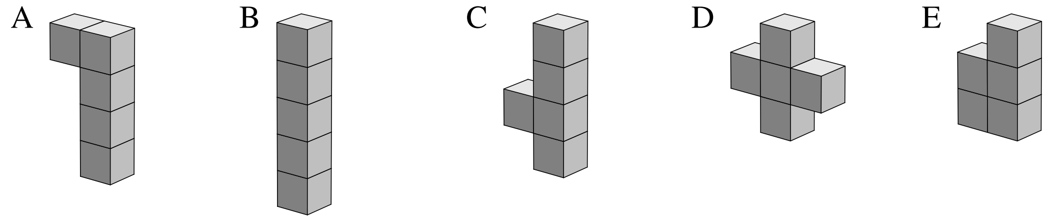
(b) 2019 thousandths,

(c) 2019 millionths,

(d) 2019 billionths.

2. Each of the five shapes shown below is made from five unit cubes.

Which has the smallest surface area?



SOLUTION **E**

Each cube has six faces, and so between them the five unit cubes have 30 faces.

The surface area of a shape is determined by how many of the faces of each cube are on the outside of the shape. When two of the cubes touch, two faces are hidden and so don't form part of the surface of the shape.

Therefore the shape with the smallest surface area is the shape where the largest number of pairs of cubes touch each other.

It can be seen that in each of the shapes A, B, C, and D the number of pairs of faces that touch is 4, but in the shape of option E, there are 5 pairs of cubes that touch.

Hence the shape that has the smallest surface area is the shape of option E.

FOR INVESTIGATION

2.1 Find the surface area of each of the five shapes.

2.2 Is there a shape built from the five unit cubes that has a smaller surface area than the shape of option E?

2.3 What is the smallest surface area of a shape made from six unit cubes?

3. There are 120 000 red squirrels living in Scotland. This represents 75% of their total UK population.

How many more red squirrels live in Scotland than live in the remainder of the UK?

A 30 000 B 40 000 C 60 000 D 80 000 E 90 000

Solution **D**

Because 75% of the red squirrels live in Scotland, 25% live in the remainder of the UK.

Now 25% is one third of 75%. Therefore the number of squirrels in the remainder of the UK is $\frac{1}{3} \times 120\,000 = 40\,000$.

Therefore there are $120\,000 - 40\,000$, that is, 80 000, more red squirrels living in Scotland than in the remainder of the UK.

For investigation

3.1 Based on the numbers given in this question, how many red squirrels are there in the UK?

3.2 If the number of red squirrels in Scotland were 80% of the total UK population of 120 000, how many more red squirrels would live in Scotland than in the rest of the UK?

3.3 It is estimated that only one in eight of the UK's red squirrels live in England.

(a) What is this as a percentage?

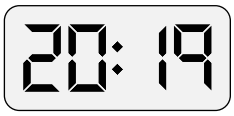
(b) Based on this estimate, how many more red squirrels live in Scotland than in England?

3.4 It is estimated that there are 110 000 fewer red squirrels in Wales than in the rest of the UK. On the basis of this estimate, what percentage of the UK's red squirrels live in Wales?

3.5 The number of humans living in the UK, to the nearest 1 million, is 67 million. To the nearest 10, how many humans are there in the UK for each red squirrel?

4. A 24-hour digital clock shows the time in hours and minutes.

How many times in one day will it display all four digits 2, 0, 1 and 9 in some order?



A 6 B 10 C 12 D 18 E 20

Solution **B**

Commentary

Because the number of arrangements of the digits which correspond to a time is quite small, the most straightforward method is just to list them all. This is the first method that we use.

When answering a question of this type by listing all the possibilities, it is important to list them in a systematic way, so as to be sure that no case has been left out. Here, the systematic method is to list the cases in order of the times to which they correspond.

> Our second method avoids listing all the cases by counting them in a systematic way using combinatorial methods.

Method 1

The times when the four digits 2, 0, 1 and 9 are displayed in some order are

01:29 02:19 09:12 09:21 10:29 12:09 19:02 19:20 20:19 21:09.

We see that there are 10 times in the day when the clock displays the four given digits in some order.

Method 2

The four digits 2, 0, 1 and 9 can be arranged in order in 24 ways, but not all these arrangements correspond to actual times. The digit 9 can occur only in the second and fourth positions because there are only 24 hours in a day and only 60 minutes in an hour.

There is no hour 29, so if the digit 9 is in the second position, the first digit can only be 0 or 1. In each of these two cases, the remaining two digits can be arranged in the remaining two places in either order.

So there are $2 \times 2 = 4$ times when the digit 9 occurs in the second place.

If the digit 9 is in the fourth position, the remaining three digits can be arranged in the remaining three places in any order. Three numbers may be arranged in order in $3 \times 2 \times 1 = 6$ different ways.

So there are 6 times when the digit 9 occurs in the fourth place.

This makes a total of $4 + 6 = 10$ times in one day when the digits 2, 0, 1 and 9 are displayed in some order.

For investigation

4.1 Another 24-hour digital clock shows the time in hours, minutes and seconds.

(a) How many times a day does the clock display all the six digits 0, 1, 2, 3, 4 and 5 in some order?

(b) How many times a day does the clock display all the six digits 1, 2, 3, 4, 5 and 6 in some order?

(c) How many times a day does the clock display all the six digits 2, 3, 4, 5, 6 and 7 in some order?

(d) How many times a day does the clock display all the six digits 3, 4, 5, 6, 7 and 8 in some order?

5. The answers to the three calculations below are to be written in descending order.

$$\begin{array}{ll} X & 0.6 \times 0.5 + 0.4 \\ Y & 0.6 \times 0.5 \div 0.4 \\ Z & 0.6 \times 0.5 \times 0.4 \end{array}$$

What is the correct order?

A YZX B XYZ C XZY D YXZ E ZYX

SOLUTION **D**

We have

$$\begin{array}{ll} X & 0.6 \times 0.5 + 0.4 = 0.3 + 0.4 = 0.7, \\ Y & 0.6 \times 0.5 \div 0.4 = 0.3 \div 0.4 = 0.75, \\ Z & 0.6 \times 0.5 \times 0.4 = 0.3 \times 0.4 = 0.12. \end{array}$$

We therefore see that expression Y has the largest value and expression Z has the smallest value.

Hence the correct descending order is YXZ.

FOR INVESTIGATION

5.1 Arrange the following in increasing numerical order.

(a) $0.6 \times 0.5 + 0.4 \times 0.3$

(b) $0.6 + 0.5 \times 0.4 \times 0.3$

(c) $0.6 \div 0.5 + 0.4 \times 0.3$

(d) $0.6 \times 0.5 \times 0.4 \times 0.3$

(e) $0.6 + 0.5 + 0.4 + 0.3$

5.2 Each of the symbols $\square$, $\diamond$ and $\heartsuit$ may be replaced by one of the operations $+, -, \times$ and $\div$. Different symbols may be replaced by the same operation.

Which choices of the operations make the value of the expression

$$0.6 \square 0.5 \diamond 0.4 \heartsuit 0.3$$

as large as possible?

6. The diagram shows part of a tessellation of the plane by a quadrilateral.

Khelen wants to colour each quadrilateral in the pattern so that no two quadrilaterals that meet (even at a point) have the same colour.

What is the smallest number of colours he needs?

A 3 B 4 C 5 D 6 E 7

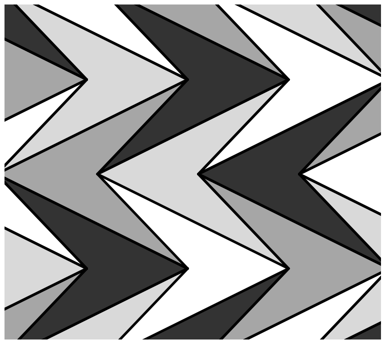
SOLUTION **B**

The quadrilaterals that meet at each vertex need to be given different colours.

Because four quadrilaterals meet at each vertex, Khelen needs at least 4 colours.

The figure shows that it is possible to colour the quadrilaterals using just 4 colours, so that no two quadrilaterals that meet at a vertex have the same colour.

Hence the smallest number of colours that Khelen needs is 4.



FOR INVESTIGATION

6.1 Suppose that quadrilaterals that meet along an edge must be coloured differently, but quadrilaterals that meet just at a point may have the same colour.

What is the smallest number of colours that Khelen would need in this case?

6.2 What is the smallest number of colours needed to colour the edges so that edges that meet at a vertex are coloured differently?

7. How many positive cubes less than 5000 end in the digit 5?

A 1 B 2 C 3 D 4 E 5

SOLUTION **B**

The cube of a positive integer ends in the digit 5 only when the positive integer itself ends in the digit 5. So we check the cubes of the positive integers that have units digit 5, until we reach a cube which is greater than 5000.

We have $5^3 = 125 < 5000$, $15^3 = 3375 < 5000$, but $25^3 = 15\,625 > 5000$.

We deduce that there are two positive cubes that end in the digit 5.

FOR INVESTIGATION

7.1 For each of the digits 0, 1, 2, 3, 4, 6, 7, 8 and 9, count the number of positive cubes less than 5000 that end with that digit.

7.2 What proportion of all the cubes in the range from 1 to 1 000 000 end in the digit 7?

8. Three consecutive positive integers less than 20 are, in ascending order, prime, even and triangular.

What is the product of these three integers?

A 6 B 60 C 990 D 1786 E 2730

SOLUTION **E**

The triangular numbers that are less than 20 are 1, 3, 6, 10 and 15.

The largest of the three consecutive positive integers is one of these numbers. It is odd as it immediately follows an even number. It is not 1 as it is the largest of three positive integers. Hence it is either 3 or 15.

Therefore the three consecutive integers are either 1, 2, 3 or 13, 14, 15.

The first number in the sequence is a prime number. This rules out the sequence 1, 2, 3 because 1 is not a prime number.

However, 13 is a prime number. So the three consecutive positive integers are 13, 14, 15. Their product is given by $13 \times 14 \times 15 = 13 \times 210 = 2730$.

It is important to remember that 1 is not regarded as a prime number. This is only a convention, but it is a standard convention.

As Vicky Neale says in her recent book, *Closing the Gap: the quest to understand prime numbers*, Oxford University Press, 2017, "In case you're wondering, I'll mention that 1 is not prime. That's not because of some interesting philosophical point, or because it's only got one number that divides into it rather than two, or anything like that. Definitions (such as that of a prime number) don't get handed to mathematicians on stone tablets. Rather, part of the job of mathematicians is to make good definitions, ones that lead to interesting mathematics, and it turns out to be better to define 1 not to be prime."

FOR INVESTIGATION

8.1 Is it possible to have three consecutive positive integers that are all primes?

8.2 Is it possible to have four consecutive integers, all greater than 2, such that three of them are prime?

8.3 Is it possible to have four consecutive positive integers none of which is prime?

9. What is the value of $\big((7 - 6 \times (-5)\big) - 4 \times (-3) \div (-2)$?

A -17 B -11 C 31 D 37 E 43

SOLUTION **C**

We have $7 - 6 \times (-5) = 7 + 30 = 37$ and $4 \times (-3) \div (-2) = -12 \div -2 = 6$.

Hence $\big((7 - 6 \times (-5))\big) - 4 \times (-3) \div (-2) = 37 - 6 = 31$.

10. A recent report about the amount of plastic created in the last 65 years stated that the 8.3 billion tonnes produced is as heavy as 25 000 Empire State Buildings in New York or a billion elephants.

On that basis, how many elephants have the same total weight as the Empire State Building?

A 4000 B 40 000 C 400 000 D 4 000 000 E 40 000 000

Solution **B**

A billion is 1 000 000 000. So we are told that 1 000 000 000 elephants weigh the same as 25 000 Empire State Buildings.

Therefore the number of elephants that have the same weight as one Empire State Building is given by

$$\frac{1\,000\,000\,000}{25\,000} = \frac{1\,000\,000}{25} = 40\,000.$$

For investigation

10.1 Based on the information given in the question, what is the weight of the Empire State Building, and what is the weight of a typical elephant?

11. Which of the following is equal to $\dfrac{3^9}{9^3}$?

A 3 B 9 C 27 D 81 E 243

Solution **C**

Because $9 = 3^2$, we have $9^3 = (3^2)^3 = 3^6$. Therefore,

$$\frac{3^9}{9^3} = \frac{3^9}{3^6} = 3^{9-6} = 3^3 = 27.$$

For investigation

11.1 Our solution uses the facts that $(3^2)^3 = 3^6$ and $\dfrac{3^9}{3^6} = 3^{9-6}$. These are special cases of the general results that, for all positive integers m and n, and all real numbers a, we have

(a) $(a^m)^n = a^{mn}$, and (b) for $m > n$, $\dfrac{a^m}{a^n} = a^{m-n}$.

Explain why these are true.

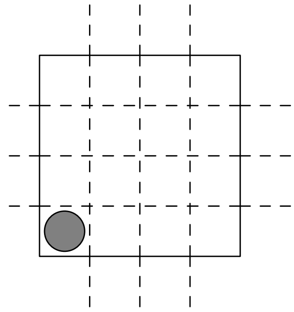
11.2 Find the integer n such that $\dfrac{27^{81}}{81^{27}} = 3^n$.

11.3 Find the integer n such that $\dfrac{16^{256}}{256^{16}} = 2^n$.

12. The game of *Rorrim2* is played on a 4×4 board, starting with a counter in one corner, as shown.

At each turn, the player moves the counter to a cell that is the reflection of its current cell in one of the six dashed lines.

How many cells could the counter occupy after precisely three turns?



A 4 B 6 C 8 D 12 E 16

Solution **C**

In the figures below we have shaded alternate squares of the board to make it easier to see what is going on.

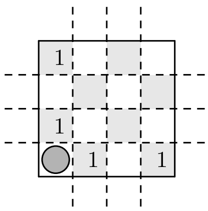


Figure 1

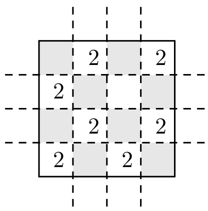


Figure 2

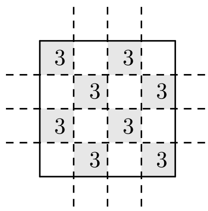


Figure 3

In Figure 1 we have shown all the cells that the counter could occupy after one turn. These positions are found by considering all the possible moves that the player can make when the counter starts off in the bottom left hand cell of the board.

Figure 2 shows all the cells that the counter could occupy after two turns. These positions are found by considering, in turn, all the player's possible moves when the counter is in one of the positions marked '1' in Figure 1.

In a similar way Figure 3 shows all the cells that the counter could occupy after three turns.

From Figure 3 we see that there are 8 cells which the counter could occupy after precisely three turns.

For investigation

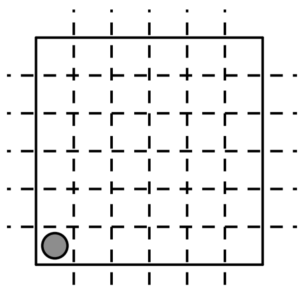
12.1 How many cells could the counter occupy after at most three turns?

12.2 How many cells could the counter occupy after precisely four turns.

12.3

(a) Now consider the same game played on a 6×6 board. For $k = 1, 2, \ldots$ determine how many cells the counter could occupy after k turns.

(b) Which is the least k such that each of the cells could be occupied after at most k turns?



13. Megan writes down a list of five numbers. The mean of her first three numbers is -3. The mean of her first four numbers is 4. The mean of her first five numbers is -5.

What is the difference between her fourth number and her fifth number?

A 66 B 55 C 44 D 33 E 22

SOLUTION **A**

Because the mean of Megan's first three numbers is -3, the sum of her first three numbers is $3 \times -3 = -9$.

Similarly, the sum of her first four numbers is $4 \times 4 = 16$, and the sum of her first five numbers is $5 \times -5 = -25$.

It follows that Megan's fourth number is $16 - (-9) = 25$ and her fifth number is $-25 - 16 = -41$.

Therefore the difference between her fourth number and her fifth number is $25 - (-41) = 66$.

FOR INVESTIGATION

13.1 Find the sequence of four numbers for which the first is 1, the mean of the first two is 2, the mean of the first three is 3, and the mean of the first four is 4.

13.2 An infinite sequence of numbers has the property that, for each positive integer n, the mean of the first n numbers in the sequence is n.

Find a formula for the nth number in the sequence.

13.3 An infinite sequence of numbers has the property that, for each positive integer n, the mean of the first n numbers in the sequence is n^2.

Find a formula for the nth number in the sequence.

14. There are four people, some of whom always tell the truth. The others always lie.
The first person said, “An odd number of us always tell the truth”.
The second person said, “An even number of us always tell the truth”.
The third person said, “A prime number of us always tell the truth”.
The fourth person said, “A square number of us always tell the truth”.

How many of these four people were telling the truth?

A 0 B 1 C 2 D 3 E 4

Solution **C**

> Commentary
>
> Our first method involves thinking about what the different people say logically.
>
> The second method indicates a more computational approach.
>
> The first method leads to a shorter solution. The second method might be of more use in a more complicated problem of this type.

Method 1

We note first that if a person is telling the truth, then they always tell the truth. Similarly, if they are telling a lie, then they always tell lies.

An integer is either even or odd, but not both. Therefore one of the first two people always tells the truth and one always lies.

A square is not a prime, so at most one of the last two people always tells the truth.

We deduce from this that the number of the people who always tell the truth is either one or two.

However, if just one person always tells the truth and the other three always lie, both the first and fourth people are telling the truth, as 1 is both odd and a square. Hence two of the people always tell the truth. This is contradiction. So this case doesn’t occur.

If just two people always tell the truth, then the second and third people are telling the truth, because 2 is even and a prime, and the other two are not telling the truth. So in this case there is no contradiction.

We deduce that 2 is the number of people who were telling the truth.

Method 2

In columns two to five of the table below we have systematically listed all 16 different combinations of the people who always tell the truth, indicated by T, and who always lie, indicated by L.

We use P1, P2, P3, P4 to indicate the people.

In the sixth column we give the number of truth tellers according to the information in the earlier columns.

The last 4 columns indicate whether or not the four people are telling the truth based on what they say and the number of truth tellers as given in the sixth column. For example, in row 16, because 0 is even and a square, in this row it is just P2 and P4 who are truth tellers.

row no.	P1	P2	P3	P4	no. of truth tellers	P1	P2	P3	P4
1	T	T	T	T	4	L	T	L	T
2	T	T	T	L	3	T	L	T	L
3	T	T	L	T	3	T	L	T	L
4	T	T	L	L	2	L	T	T	L
5	T	L	T	T	3	T	L	T	L
6	T	L	T	L	2	L	T	T	L
7	T	L	L	T	2	L	T	T	L
8	T	L	L	L	1	T	L	L	T
9	L	T	T	T	3	T	L	T	L
10	L	T	T	L	2	L	T	T	L
11	L	T	L	T	2	L	T	T	L
12	L	T	L	L	1	T	L	L	T
13	L	L	T	T	2	L	T	T	L
14	L	L	T	L	1	T	L	L	T
15	L	L	L	T	1	T	L	L	T
16	L	L	L	L	0	L	T	L	T

If, in a given row, the information in the last four columns is not the same as that in columns two to five, the situation in that row cannot occur.

From the table we see that the only situation that can occur is that described in row 10. We deduce that 2 of the people were telling the truth.

For investigation

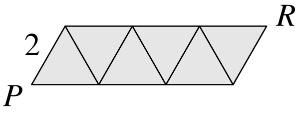
14.1 For each person there are 2 possibilities. They are either a truth teller or a liar. Why does it follow that we needed to have 16 rows in the above table? How many rows would be needed if there were 5 people?

14.2 Columns 2 to 5 in the above table have been set out systematically to make sure that none of the combinations has been excluded. Can you describe the system that has been used?

14.3 Suppose that in the first four columns you replace T by 1 and L by 0. What do you then notice?

15. The diagram shows six congruent equilateral triangles, of side-length 2, placed together to form a parallelogram.

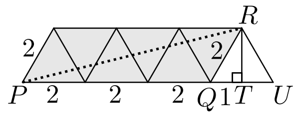
What is the length of PR?



A $2\sqrt{13}$ B 7 C $6\sqrt{3}$ D 9 E $7\sqrt{3}$

Solution **A**

The point Q is a vertex of the parallelogram, as shown. We extend the line PQ to the point U so that QU has length 2.



It can be seen that RQU is an equilateral triangle with side length 2.

We let T be the foot of the perpendicular from R to QU.

It can be seen that the triangles RTQ and RTU are congruent. It follows that $QT = TU = 1$.

By Pythagoras' Theorem, applied to the right-angled triangle RTQ, we have $RQ^2 = RT^2 + TQ^2$. It follows that $RT = \sqrt{RQ^2 - TQ^2} = \sqrt{2^2 - 1^2} = \sqrt{3}$.

We see that $PT = 7$.

By Pythagoras' Theorem, applied to the right-angled triangle RTP, we have $PR^2 = PT^2 + RT^2 = 7^2 + \sqrt{3}^2 = 49 + 3 = 52$. Therefore $PR = \sqrt{52} = 2\sqrt{13}$.

For investigation

15.1 Prove that the triangle RQU is equilateral.

15.2 Prove that the triangles RTQ and RTU are congruent.

15.3

The diagram shows $2n$ congruent equilateral triangles, of side length 2, placed together to form a parallelogram, where n is some positive integer.

(a) Find the length of PS.

(b) Prove that there is no positive integer n for which the length of PS is an integer.

16. Two numbers x and y have a product which is equal to their sum.

Which of these expressions gives x in terms of y?

A $\dfrac{y}{y-1}$ B $\dfrac{y}{y+1}$ C $\dfrac{y+1}{y}$ D $\dfrac{y-1}{y}$ E $\dfrac{y^2}{y+1}$

Solution **A**

Because the product of x and y is the same as their sum,

$$xy = x + y.$$

By subtracting x from both sides, we see that this equation is equivalent to

$$xy - x = y.$$

This last equation is equivalent to

$$x(y-1) = y.$$

We should now like to divide both sides of this last equation by $y - 1$, but we need first to check that we are not dividing by an expression that could have the value 0.

We see that if y were equal to 1, the equation $xy = x + y$ would become $x = x + 1$, which is impossible. We deduce that y cannot have the value 1 and therefore $y - 1$ cannot be 0.

It therefore follows that

$$x = \frac{y}{y-1}.$$

17. Which of these is equal to $0.\dot{8} + 0.0\dot{7}$?

A $0.8\dot{7}$ B 0.88 C $0.9\dot{5}$ D $0.9\dot{6}$ E $0.9\dot{8}$

Solution **D**

Commentary

The difficulty with this question is that $0.\dot{8}$ and $0.0\dot{7}$ are *recurring* decimals, so that if we could write them in full they would each have an infinite number of digits.

When we add numbers with only a finite number of places, the method is to start at the right, adding the corresponding digits and keeping track of the need to carry a 1 to the next place. For example, when adding 1.234 and 5.678 there are two carries (from the units column to the tens column, and from the tens column to the hundreds column):

$$\begin{array}{r} 1.234 \\ +\ \underline{5.678} \\ \underline{6.912} \end{array}$$

> However, with infinite decimal expansions, there is no last place on the right where we can begin the addition:
>
> $$\begin{array}{r} 0.8888888..... \\ +\ \underline{0.0777777.....} \\ \underline{?.???????.....} \end{array}$$
>
> We can get over this difficulty in two ways. We can convert the recurring decimals into fractions, add the fractions, and then convert the answer back into a decimal. Alternatively, we can replace the infinite decimal expansions by finite approximations.

METHOD 1

We have $0.\dot{8} = \frac{8}{9}$ and $0.0\dot{7} = \frac{7}{90}$. Therefore

$$\begin{aligned} 0.\dot{8} + 0.0\dot{7} &= \frac{8}{9} + \frac{7}{90} \\ &= \frac{80}{90} + \frac{7}{90} \\ &= \frac{87}{90} \\ &= \frac{81}{90} + \frac{6}{90} \\ &= \frac{9}{10} + \frac{2}{30} \\ &= 0.9 + 0.0\dot{6} \\ &= 0.9\dot{6}. \end{aligned}$$

METHOD 2

The standard method for adding decimals with a finite number of decimal places gives us

$$\begin{aligned} 0.88 + 0.07 &= 0.95 \\ 0.888 + 0.077 &= 0.965 \\ 0.8888 + 0.0777 &= 0.9665 \\ 0.88888 + 0.07777 &= 0.96665 \qquad \text{and so on.} \end{aligned}$$

We therefore see that, 'in the limit',

$$0.\dot{8} + 0.0\dot{7} = 0.9\dot{6}.$$

FOR INVESTIGATION

17.1 Express as a single decimal (a) $0.\dot{6} + 0.\dot{7}$, (b) $0.\dot{7} + 0.0\dot{6}$ and (c) $0.\dot{6}\dot{7} + 0.0\dot{7}\dot{6}$.

18. Two numbers x and y are such that $x + y = \frac{2}{3}$ and $\frac{x}{y} = \frac{2}{3}$.

What is the value of $x - y$?

A $-\frac{2}{3}$ B $-\frac{2}{15}$ C $\frac{2}{25}$ D $\frac{2}{5}$ E $\frac{2}{3}$

Solution **B**

We avoid fractions by multiplying both sides of the first equation by 3, and both sides of the second equation by $3y$. This gives

$$3x + 3y = 2$$

and

$$3x = 2y.$$

Substituting from the second equation into the first,

$$2y + 3y = 2,$$

that is,

$$5y = 2.$$

Therefore

$$y = \frac{2}{5}.$$

Hence, as $x + y = \frac{2}{3}$,

$$\begin{aligned} x &= \frac{2}{3} - \frac{2}{5} \\ &= \frac{10 - 6}{15} \\ &= \frac{4}{15} \end{aligned}$$

Therefore

$$\begin{aligned} x - y &= \frac{4}{15} - \frac{2}{5} \\ &= \frac{4 - 6}{15} \\ &= -\frac{2}{15}. \end{aligned}$$

For investigation

18.1 The two equations given in the question may be written as $3x + 3y = 2$ and $3x - 2y = 0$. Show that

$$x - y = -\tfrac{1}{15}(3x + 3y) + \tfrac{2}{5}(3x - 2y).$$

Use this to calculate the value of $x - y$ without the need to find x and y separately.

19. Which of these expressions has the largest value?

A $\frac{1}{2}$ B $\frac{1}{3}+\frac{1}{4}$ C $\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$

D $\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}$ E $\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}$

SOLUTION **E**

Because $\frac{1}{3} > \frac{1}{4}$, we have

$$\frac{1}{3}+\frac{1}{4} > \frac{1}{4}+\frac{1}{4} = \frac{1}{2}.$$

Because $\frac{1}{5} > \frac{1}{6}$, we have

$$\frac{1}{4}+\frac{1}{5}+\frac{1}{6} > \frac{1}{4}+\frac{1}{6}+\frac{1}{6} = \frac{1}{4}+\frac{1}{3} = \frac{1}{3}+\frac{1}{4}.$$

Because $\frac{1}{7} > \frac{1}{8}$, we have

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > \frac{1}{5}+\frac{1}{6}+\frac{1}{8}+\frac{1}{8} = \frac{1}{5}+\frac{1}{6}+\frac{1}{4} = \frac{1}{4}+\frac{1}{5}+\frac{1}{6}.$$

Finally, because $\frac{1}{9} > \frac{1}{10}$, we have

$$\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10} > \frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{10}+\frac{1}{10} = \frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{5} = \frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}.$$

We therefore see that the values of the expressions given in the question increase as we go from option A to option E. Hence the expression with the greatest value is that given by option E.

FOR INVESTIGATION

19.1 Show that

(a) $\frac{1}{2}+\frac{1}{3}+\frac{1}{4} > 1.$

(b) $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > \frac{3}{2}.$

(c) $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16} > 2.$

19.2 Prove that for each positive integer n,

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^{2^n}-1}+\frac{1}{2^{2^n}} > 1+\frac{n}{2}.$$

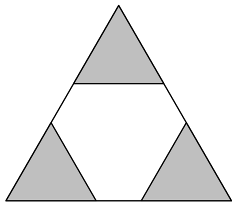
Note: It follows that the infinite series (called the *harmonic series*)

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\ldots$$

does not converge to a finite limit.

20. Three equilateral triangles with sides of length 1 are shown shaded in a larger equilateral triangle. The total shaded area is half the area of the larger triangle.

What is the side-length of the larger equilateral triangle?



A $\sqrt{5}$ B $\sqrt{6}$ C $\frac{5}{2}$ D $\frac{3\sqrt{3}}{2}$ E $1+\sqrt{3}$

Solution **B**

We let s be the side length of the larger equilateral triangle.

The ratio of the area of similar triangles (or, indeed, other pairs of similar figures) is the same as the ratio of the squares of the lengths of their corresponding sides.

Therefore the area of the larger equilateral triangle is s^2 times the area of each of the three equilateral triangles with side length 1.

Since the total shaded area is half the area of the larger triangle, the area of the larger equilateral triangle is twice the total area of the three smaller equilateral triangles, and hence is six times the area of any one of these triangles.

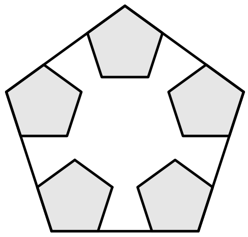
It follows that $s^2 = 6$, and therefore $s = \sqrt{6}$.

For investigation

20.1 Five regular pentagons, each with side length 1, are shown shaded in a larger regular pentagon.

The total shaded area is half the area of the larger regular pentagon.

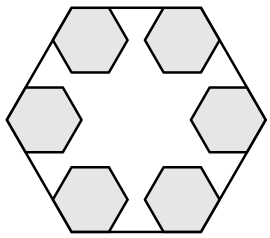
What is the side-length of the larger regular pentagon?



20.2 Six regular hexagons, each with side length 1, are shown shaded in a larger regular hexagon.

The total shaded area is half the area of the larger regular hexagon.

What is the side-length of the larger regular hexagon?



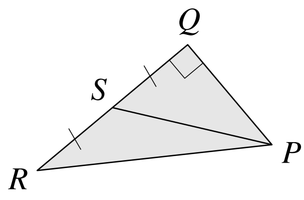
20.3 Find, in terms of n, the side length of a regular polygon with n sides which has twice the total area of n regular polygons, each of which has n sides of length 1.

20.4 The above solution uses the fact that the ratio of the areas of similar triangles is the same as the ratio of the squares of the lengths of their corresponding sides. Prove that this is correct.

21. The diagram shows a right-angled triangle PQR. The point S is the midpoint of the side QR and $\tan \angle QPR = \frac{3}{2}$.

What is the value of $\sin \angle QPS$?

A $\dfrac{1}{\sqrt{3}}$ B $\dfrac{1}{\sqrt{2}}$ C $\dfrac{1}{2}$ D $\dfrac{3}{5}$ E $\dfrac{4}{5}$



Solution **D**

We let QS have length s. Then, as S is the midpoint of QR, the length of QR is $2s$.

Now $\tan \angle QPR = \dfrac{QR}{QP}$. Therefore, by the information given in the question, we have

$$\frac{2s}{QP} = \frac{QR}{QP} = \tan \angle QPR = \frac{3}{2}.$$

and therefore

$$QP = \frac{2}{3} \times 2s = \frac{4}{3}s.$$

Hence, by Pythagoras' Theorem, applied to the right-angled triangle SQP,

$$SP^2 = QS^2 + QP^2 = s^2 + \left(\frac{4}{3}s\right)^2 = s^2 + \frac{16}{9}s^2 = \left(1 + \frac{16}{9}\right)s^2 = \frac{25}{9}s^2.$$

If follows that

$$SP = \frac{5}{3}s.$$

Therefore

$$\sin \angle QPS = \frac{QS}{PS} = \frac{s}{\frac{5}{3}s} = \frac{3}{5}.$$

Note

You will see that we could have avoided almost all of the fractions in this solution by letting the length of QS be $3s$ at the start of the question. But this is being wise after the event, and given the limited time that you have for the IMC, you could be forgiven for not looking ahead in this way.

For investigation

21.1 Suppose that QS has length $3s$. What in this case would be the length of PS?

21.2 What are the values of $\sin \angle SQP$ and $\cos \angle SQP$?

21.3 Evaluate $\cos^2 \angle SQP + \sin^2 \angle SQP$.

Note: $\cos^2 t$ and $\sin^2 t$ are the standard ways of writing $(\cos t)^2$ and $(\sin t)^2$.

21.4 Does your answer to 21.3 generalize? If so, can you prove this?

22. Four of the following six-digit integers are always divisible by 7, regardless of the values of the digits P and Q.

Which of the following is not neccesarily a multiple of 7?

A '$PQQ\,PQQ$' B '$PQP\,QPQ$' C '$QPQ\,PQP$' D '$PPP\,PPP$' E '$PPP\,QQQ$'

SOLUTION **E**

COMMENTARY

The key facts that we use in this question are that both 1001 and 10 101 are multiples of 7.

In fact, $1001 = 7 \times 11 \times 13$, a factorization which it is well worth remembering, and $10\,101 = 3 \times 7 \times 13 \times 37$, which is not quite so memorable.

We also need to recall that the notation '$PQRSTU$', where P, Q, R, S, T and U are digits, represents the number $100\,000P + 10\,000Q + 1000R + 100S + 10T + U$.

We see that

$$\begin{aligned} \text{`}PQQPQQ\text{'} &= 1001 \times \text{`}PQQ\text{'},\\ \text{`}PQPQPQ\text{'} &= 10101 \times \text{`}PQ\text{'},\\ \text{`}QPQPQP\text{'} &= 10101 \times \text{`}QP\text{'}, \end{aligned}$$

and

$$\text{`}PPPPPP\text{'} = 1001 \times \text{`}PPP\text{'}.$$

Now $1001 = 7 \times 11 \times 13$ and $10\,101 = 3 \times 7 \times 13 \times 37$. It follows that all of '$PQQPQQ$', '$PQPQPQ$', '$QPQPQP$' and '$PPPPPP$' are necessarily multiples of 7.

In the context of the IMC we can now deduce that the remaining option, '$PPPQQQ$' is not necessarily a multiple of 7. [Exercise 22.3 below asks you to check this.]

FOR INVESTIGATION

22.1 Check that $1001 = 7 \times 11 \times 13$ and that $10\,101 = 3 \times 7 \times 13 \times 37$.

22.2 Check that

$$\begin{aligned} &(a) \quad \text{`}PQQPQQ\text{'} = 1001 \times \text{`}PQQ\text{'}\\ &(b) \quad \text{`}PQPQPQ\text{'} = 10101 \times \text{`}PQ\text{'}\\ &(c) \quad \text{`}QPQPQP\text{'} = 10101 \times \text{`}QP\text{'}\\ &(d) \quad \text{`}PPPPPP\text{'} = 1001 \times \text{`}PPP\text{'} \end{aligned}$$

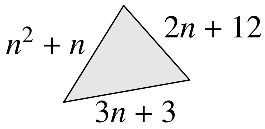
22.3 Give an example of digits P and Q such that '$PPPQQQ$' is not a multiple of 7.

22.4 Is it possible for '$PPPQQQ$' to be a multiple of 7?

22.5 Which primes are factors of all integers of the form '$PPPQQQ$'?

23. The diagram shows a triangle with sides $n^2 + n$, $2n + 12$ and $3n + 3$.

What is the sum of all the values of n for which the triangle is isosceles?



A 7 B 9 C 12 D 13 E 16

SOLUTION **A**

COMMENTARY

In this solution we use the symbol $\Leftrightarrow$ to mean *is equivalent to*.

For the triangle to be isosceles two of its sides need to be equal. We therefore equate, in turn, the three pairs of formulas for the side lengths.

First we have

$$\begin{aligned} n^2 + n = 2n + 12 &\Leftrightarrow n^2 - n - 12 = 0 \\ &\Leftrightarrow (n-4)(n+3) = 0 \\ &\Leftrightarrow n = 4 \text{ or } n = -3. \end{aligned}$$

Now, when $n = 4$ the sides of the triangle are 20, 20 and 15, which is possible for a triangle. If n were -3, the side lengths would be 6, 6 and -6, which is impossible, because the side length of a triangle is a positive number. So 4 is a possible value of n, but -3 is not.

Second,

$$\begin{aligned} n^2 + n = 3n + 3 &\Leftrightarrow n^2 - 2n - 3 = 0 \\ &\Leftrightarrow (n-3)(n+1) = 0 \\ &\Leftrightarrow n = 3 \text{ or } n = -1 \end{aligned}$$

When $n = 3$ the sides of the triangles are 12, 18 and 12, which is possible for a triangle. If n were -1 the sides lengths would be 0, 12 and 0, which is impossible. So 3 is also a possible value of n, but -1 is not.

Finally,

$$3n + 3 = 2n + 12 \Leftrightarrow n = 9$$

If n were 9 the side lengths would be 90, 30 and 30. This is impossible for a triangle, because the length of one side of a triangle cannot be greater than the sum of the lengths of the other two sides and $90 > 30 + 30$.

Therefore, the values of n for which the triangle is isosceles are 4 and 3. The sum of these values is $4 + 3 = 7$.

24. When 5655 is divided by a two-digit positive integer N, the remainder is 11. When 5879 is divided by the same positive integer N, the remainder is 14.

What is the sum of the digits of N?

A 6 B 7 C 8 D 9 E 10

SOLUTION **C**

Because 5655 has remainder 11 when divided by N, it follows that $5644 = 5655 - 11$ is a multiple of N.

Similarly, because 5879 has remainder 14 when divided by N, it follows that $5879 - 14 = 5865$ is also a multiple of N.

Because 5644 and 5865 are both multiples of N, so also is their difference.

Now $5865 - 5644 = 221$ and therefore 221 is a multiple of N. In other words, N is a two-digit factor of 221.

When we factorize 221 into primes, we obtain $221 = 13 \times 17$. Therefore N is either 13 or 17.

It can be checked that 17 is a factor of both 5644 and 5865, but neither has 13 as a factor.

Hence $N = 17$. Therefore the sum of the digits of N is $1 + 7 = 8$.

FOR INVESTIGATION

24.1 Check that

(a) the remainder when 5655 is divided by 17 is 11, and the remainder when 5879 is divided by 17 is 14,

(b) 17 is a factor of both 5644 and 5865,

(c) neither 5644 nor 5865 has 13 as a factor.

24.2 When 7000 is divided by a three-digit positive integer N, the remainder is 99.

When 5000 is divided by the same positive integer N, the remainder is 56.

What is the value of N?

24.3 When 5000 is divided by a positive integer N, the remainder is 9.

When 1000 is divided by the same positive integer N, the remainder is 8.

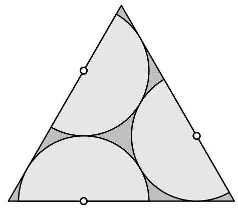
What is the value of N?

24.4 Prove that if the integers j are k are both divisible by the integer N, then both $j + k$ and $j - k$ are divisible by N.

25. The diagram shows three touching semicircles with radius 1 inside an equilateral triangle, which each semicircle also touches. The diameter of each semicircle lies along a side of the triangle.

What is the length of each side of the equilateral triangle?

A 3 B $\frac{13}{4}$ C $\frac{3}{2}\sqrt{3}$ D $2\sqrt{3}$ E 4

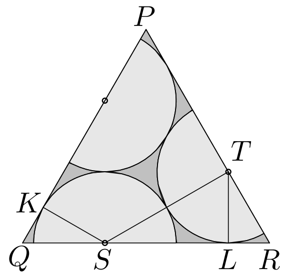


Solution **D**

Let P, Q and R be the vertices of the equilateral triangle.

Let S, T be the points on QR, RP, respectively, that are the centres of two of the semicircles.

Let K be the point where the semicircle with centre S touches PQ, and let L be the point where the semicircle with centre T touches QR.



Because the tangent to a circle is perpendicular to the radius of the circle through the point where they touch, both $\angle QKS = 90°$ and $\angle RLT = 90°$.

We leave it as an exercise to show that, because the semicircles with centres S and T touch, the line ST joining their centres passes through the point where they touch. It follows that ST has length 2.

We also leave it as an exercise to deduce that $\angle STR = 90°$.

Because they are the angles of an equilateral triangle $\angle RQP = \angle PRQ = 60°$.

Because the triangles SKQ and STR each have angles 60° and 90°, and the sum of the angles in a triangle is 180°, they also each have a third angle of 30°. Because these triangles have the same angles they are similar.

Therefore, because the triangles SKQ and STR are similar, and ST is twice the length of SK, the length of SR is twice the length of QS.

It follows that the length of QR is three times the length of QS.

From the right-angled triangle QKS, we have

$$\frac{KS}{QS} = \sin 60° = \frac{\sqrt{3}}{2}.$$

Hence

$$QS = \frac{2}{\sqrt{3}} \times KS = \frac{2}{\sqrt{3}} \times 1 = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}.$$

It follows that

$$QR = 3 \times \frac{2}{3}\sqrt{3} = 2\sqrt{3}.$$

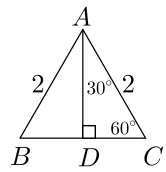
We conclude that the length of each side of the equilateral triangle is $2\sqrt{3}$.

For investigation

25.1 Show that, because the semicircles with centres S and T touch, the line ST passes through the point where they touch. Deduce that ST has length 2.

25.2 Deduce, from the result of Exercise 25.1, that $\angle STR = 90°$.

25.3 Let ABC be an equilateral triangle with sidelength 2. Let D be the point where the perpendicular from A to BC meets BC.



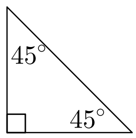
(a) Prove that D is the midpoint of BC.

(b) Use the right-angled triangle ADC to show that $\sin 60° = \dfrac{\sqrt{3}}{2}$ and $\tan 60° = \sqrt{3}$.

(c) Find the values of $\cos 60°$, $\sin 30°$, $\cos 30°$ and $\tan 30°$.

If you remember that a triangle with angles 30°, 60° and 90° forms half of an equilateral triangle, you can readily work out the values of $\cos 30°$, $\sin 30°$, tan 30, $\cos 60°$, $\sin 60°$ and $\tan 60°$, without the need to remember any of these values.

25.4 Use a right-angled triangle in which the other two angles are both 45° to find the values of



(a) $\sin 45°$,

(b) $\cos 45°$,

(c) $\tan 45°$.

Likewise, if you remember the triangle shown in this exercise, you don't need to remember the values of $\cos 45°$, $\sin 45°$ and $\tan 45°$.

25.5 Find the ratio of the total area of the three semicircles to the area of the equilateral triangle.