United Kingdom
Mathematics Trust

Intermediate Mathematical Challenge

Thursday 7 February 2019

For reasons of space, these solutions are necessarily brief.

There are more in-depth, extended solutions available on the UKMT website, which include some exercises for further investigation:

www.ukmt.org.uk

1. B The value of 2019 tenths is $2019 \times \frac{1}{10} = 201.9$

2. E Before any of the cubes are joined, the total surface area of the five cubes is equal to $5 \times 6 = 30$. Whenever two faces are joined, the total surface area decreases by two. Figures A to D all contain four joins, but figure E has five joins and therefore has the smallest surface area, namely 20.

3. D First note that three quarters of the UK red squirrels live in Scotland. So one quarter of the UK red squirrels live in the remainder of the UK and the number of such squirrels is therefore one third of the number of red squirrels living in Scotland, which is $\frac{1}{3} \times 120\,000 = 40\,000$.

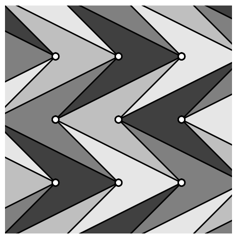
So the required difference equals $120\,000 - 40\,000 = 80\,000$.

4. B In order from 00:00, the times when the digits 2, 0, 1, 9 appear in some order are 01:29, 02:19, 09:12, 09:21, 10:29, 12:09, 19:02, 19:20, 20:19 and 21:09. So, in total, there are ten such times.

5. D The answers to the three calculations are as follows:

X $\quad 0.6 \times 0.5 + 0.4 = 0.3 + 0.4 = 0.7$;
Y $\quad 0.6 \times 0.5 \div 0.4 = 0.3 \div 0.4 = 3 \div 4 = 0.75$;
Z $\quad 0.6 \times 0.5 \times 0.4 = 0.3 \times 0.4 = 0.12$.
So the correct order is YXZ.

6. B At each of the points indicated by a dot, four quadrilaterals meet, so that at least four colours are needed.
The diagram shows that four colours is possible, so the minimum number of colours Khelen needs is four.



7. B For the cube of a positive integer to end in a 5, the integer itself must end in a 5. We note that $5^3 = 125$ and that $15^3 = 3375$. However $25^3 > 20^3 = 8000$ and so $25^3 > 5000$.
Therefore there are only two positive cubes less than 5000 which end in the digit 5, namely 125 and 3375.

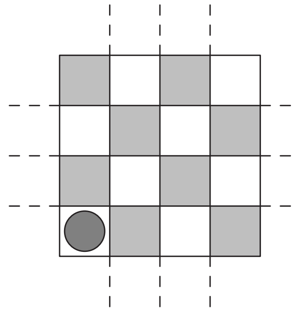
8. E First note that the triangular numbers less than 20 are 1, 3, 6, 10, 15. As we are looking for a sequence of consecutive integers which are respectively prime, even, and triangular, the required triangular number is odd. So it is 3 or 15. It is not 3 as 1 is not prime, but it is 15 as 13 is prime. So the required product = $13 \times 14 \times 15 = 2730$.

9. C The value of $(7 - 6 \times (-5)) - 4 \times (-3) \div (-2) = 7 + 30 + 12 \div (-2) = 37 + (-6) = 31$.

10. B We are told that the weight of 25 000 Empire State Buildings is equal to the weight of one billion elephants. So the Empire State Building weighs the same as $\frac{1\,000\,000\,000}{25\,000}$ elephants = $\frac{1\,000\,000}{25}$ elephants = 40 000 elephants.

11. C The fraction $\frac{3^9}{9^3} = \frac{3^9}{(3^2)^3} = \frac{3^9}{3^6} = 3^3 = 27$.

12. C The diagram shows the original diagram shaded as on a chess board. Note that as the counter starts on an unshaded cell then, in whichever line it is reflected, one reflection will move it into a shaded cell. Similarly, a second reflection will move the counter from a shaded cell into an unshaded cell. Finally, a third reflection will move the counter from an unshaded cell into a shaded cell. It is left to the reader to show that it is possible to reach each of the shaded cells in precisely three reflections . Hence the number of cells the counter could occupy after three moves is equal to the number of shaded cells, that is 8.

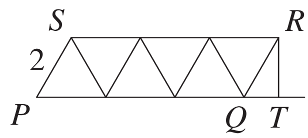


13. A From the given information, the sum of Megan's first three numbers is -9, the sum of her first four numbers is 16, and the sum of her first five numbers is -25.

Therefore the fourth number is $16 - (-9) = 25$, and the fifth number is $-25 - 16 = -41$. The difference between them is $25 - (-41) = 66$.

14. C Since no integer is both odd and even, exactly one of the first two speakers was telling the truth. No square is also prime; so at most one of the third and fourth speakers was telling the truth. Hence the number telling the truth must be one or two. If there was just one telling the truth then the statements of speakers 1 and 4 would both be true - which is a contradiction. If there were two telling the truth, then speakers 2 and 3 would be the ones telling the truth; and this is indeed two speakers.

15. A In the diagram shown, T is the foot of the perpendicular from R to PQ produced. Consider triangle QTR. Angle $QTR = 90°$ and, because the sum of angles on a straight line = 180°, angle $RQT = (180 - 2 \times 60)° = 60°$. So $\sin 60° = \frac{RT}{QR}$ and hence



$RT = QR \times \sin 60° = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$. Also $\cos 60° = \frac{QT}{QR}$, so $QT = QR \times \cos 60° = 2 \times \frac{1}{2} = 1$. Therefore $PT = 7$.

By Pythagoras' Theorem, $PR^2 = PT^2 + TR^2 = 7^2 + (\sqrt{3})^2$ $= 49 + 3 = 52$. So $PR = \sqrt{52} = \sqrt{4} \times \sqrt{13} = 2\sqrt{13}$.

16. A We are given that $xy = x + y$. So $xy - x = y$. Therefore $x(y - 1) = y$ and hence $x = \frac{y}{y-1}$.

(Note that division by $y - 1$ is allowed because if $y = 1$, then we have $x = x + 1$ which is impossible.)

17. D It may be shown that $0.\dot{1} = \frac{1}{9}$. So $0.\dot{8} = \frac{8}{9}$ and $0.0\dot{7} = \frac{7}{90}$.

Therefore $0.\dot{8} + 0.0\dot{7} = \frac{8}{9} + \frac{7}{90} = \frac{80}{90} + \frac{7}{90} = \frac{87}{90} = \frac{29}{30}$. Now $\frac{29}{3} = 9 + \frac{2}{3} = 9 + 0.\dot{6} = 9.\dot{6}$.
So $\frac{29}{30} = 0.9\dot{6}$. Hence $0.\dot{8} + 0.0\dot{7} = 0.9\dot{6}$.

(*It is left as an exercise to the reader to prove the assertion in the first sentence.*)

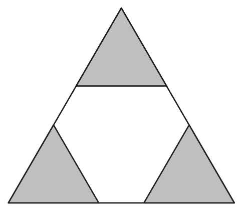
18. B From the equation $\frac{x}{y} = \frac{2}{3}$ we deduce that $3x = 2y$. We are given that $x + y = \frac{2}{3}$, so $3x + 3y = 2$.
Substituting for $3x$ gives $2y + 3y = 2$, so $y = \frac{2}{5}$.
Therefore $x - y = x + y - 2y = \frac{2}{3} - \frac{4}{5} = \frac{10}{15} - \frac{12}{15} = -\frac{2}{15}$.

19. E Note that $\frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} > \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{10} + \frac{1}{10} = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$.

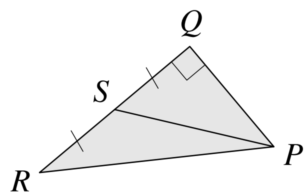
Similarly $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{5} + \frac{1}{6} + \frac{1}{8} + \frac{1}{8} = \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$. Also $\frac{1}{4} + \frac{1}{5} + \frac{1}{6} > \frac{1}{4} + \frac{1}{6} + \frac{1}{6} = \frac{1}{3} + \frac{1}{4}$.

Finally, $\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. Therefore the expression with the largest value is that in option E.

20. B Since the total shaded area is half the area of the larger equilateral triangle, the area of that triangle is six times the area of each of the shaded equilateral triangles . Now when two similar figures have side-lengths in the ratio $1 : k$, the ratio of their areas is $1 : k^2$. In this case, the ratio of the areas is $1 : 6$ so the ratio of the side-lengths is $1 : \sqrt{6}$.
Hence the the length of the sides of the larger triangle is $\sqrt{6}$ cm.



21. D Let PQ have length $4x$. Then, as $\tan \angle QPR = \frac{3}{2}$, QR has length $6x$. Therefore, as S is the midpoint of side QR, QS has length $3x$. By Pythagoras' Theorem in triangle PQS, $PS^2 = PQ^2 + QS^2 = (4x)^2 + (3x)^2 = 16x^2 + 9x^2 = 25x^2$. Hence $PS = \sqrt{25x^2} = 5x$.
So $\sin \angle QPS = \frac{QS}{PS} = \frac{3x}{5x} = \frac{3}{5}$.



22. E Note that $1001 = 7 \times 143$, so that any six-digit number of the form '$XYZ\,XYZ$' $= 1001 \times$ 'XYZ' is necessarily a multiple of 7. So options A and D are both multiples of 7.
Also, $10\,101 = 7 \times 1443$, so that any six-digit number of the form '$XYX\,YXY$' $= 10101 \times$ 'XY' is necessarily a multiple of 7. So options B and C are both multiples of 7.
However, '$PPP\,QQQ$' $= 111 \times (1000P + Q) = 3 \times 37 \times (1000P + Q)$ is not necessarily a multiple of 7. For example, when $P = 1$ and $Q = 2$, $1000P + Q = 1002 = 143 \times 7 + 1$.

23. A The triangle is isosceles when one of the following three equations is true:

$$n^2 + n = 2n + 12; \quad (1)$$
$$n^2 + n = 3n + 3; \quad (2)$$
$$2n + 12 = 3n + 3. \quad (3)$$

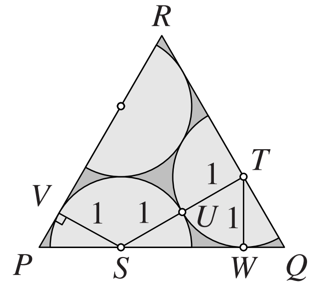
When equation (1) is true, we have $n^2 - n - 12 = 0$, so that $(n - 4)(n + 3) = 0$.
Hence either $n = 4$ or $n = -3$. However, when $n = -3$ then $3n + 3 < 0$, so that no triangle can be formed. There is, though, an isosceles triangle when $n = 4$, as the sides of the triangle are then

20, 20 and 15.
When equation (2) is true,we have $n^2 - 2n - 3 = 0$, so that $(n-3)(n+1) = 0$.
Hence either $n = 3$ or $n = -1$. However, when $n = -1$ then $3n + 3 = 0$, so that no triangle can be formed. There is, though, an isosceles triangle when $n = 3$, as the sides of the triangle are then 12, 18 and 12.
When equation (3) is true, we have $n = 9$.
But then $n^2 + n = 90$ and $3n + 3 = 30 = 2n + 12$, so that no triangle can be formed at all in this case, because $90 > 30 + 30$.
Hence the required sum is $3 + 4 = 7$.

24. C We are given that $5655 = N \times j + 11$ and $5879 = N \times k + 14$.
Subtracting the first equation from the second, we obtain, $224 = N \times (k - j) + 3$. Therefore $221 = N \times (k - j)$.

It follows that N is a positive factor of 221, so that N is equal to 1, 13, 17 or 221. But one of the remainders is 14, so N is equal to 17 or 221. However, 221 does not have two digits.
Therefore $N = 17$ and the sum of its digits is $1 + 7 = 8$.

25. D In the diagram, P, Q and R are the vertices of the triangle; S and T are both centres of a semicircle; U is a point where two semicircles touch and V and W are points where a semicircle touches the triangle. So angles PVS and TWS are right angles, since a tangent to a circle is perpendicular to the radius of that circle at the point of contact. Also, the line joining the centres of two touching semicircles passes through their point of contact, so SUT is a straight line and has length 2. Note that $\angle VPS = 60°$, and so $\angle PSV = 30°$.



Note also that $ST = 2$, $TW = 1$ and $\angle TWS$ is a right angle.
Therefore $\sin \angle WST = \dfrac{1}{2}$ and so $\angle WST = 30°$. However $\angle TQS = 60°$ and so $\angle STQ$ is a right angle.
From triangle SPV, we see that $\dfrac{SV}{PS} = \dfrac{1}{PS} = \sin 60° = \dfrac{\sqrt{3}}{2}$ and so $PS = \dfrac{2}{\sqrt{3}}$. Similarly, from triangle SQT, we see that $SQ = \dfrac{4}{\sqrt{3}}$.
Therefore $PQ = \dfrac{6}{\sqrt{3}} = \dfrac{6\sqrt{3}}{3} = 2\sqrt{3}$.