Junior Mathematical Challenge

Organised by the United Kingdom Mathematics Trust

supported by Institute and Faculty of Actuaries

Solutions and investigations

26 April 2018

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are often short so that they all fit on one sheet of paper. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some exercises for further investigation. We welcome comments on these solutions. Please send them to enquiry@ukmt.org.uk.

The Junior Mathematical Challenge (JMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the JMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. For each question we have normally given a full solution, with all steps explained, which does not use the given options. We hope that these solutions can be used as a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem, for example, in the Junior Mathematical Olympiad and similar competitions.

Comments in boxes are there for extra explanation but are not the sort of thing that should be included in full written solutions.

Enquiries about the Junior Mathematical Challenge should be sent to:

JMC, UKMT, School of Mathematics Satellite, University of Leeds, Leeds LS2 9JT

☎ 0113 343 2339 enquiry@ukmt.org.uk www.ukmt.org.uk

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
C	D	A	E	A	D	B	D	C	C	D	A	D	E	D	E	B	B	C	B	D	B	A	B	D

1. What is the value of $(222 + 22) \div 2$?

A 111 B 112 C 122 D 133 E 233

Solution **C**

We have

$$(222 + 22) \div 2 = 222 \div 2 + 22 \div 2 = 111 + 11 = 122.$$

2. A train carriage has 80 seats. On my journey I noticed that all the seats in my carriage were taken and 7 people were standing.

At Banbury, 9 people left the carriage, 28 people entered it and all the seats were taken.

How many people now had no seat?

A 0 B 7 C 16 D 26 E 35

Solution **D**

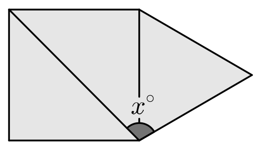
At Banbury 9 people left the carriage and 28 entered it. Therefore after the stop at Banbury there were $28 - 9$, that is, 19, more people in the train carriage than before. Therefore, as all the seats were taken, the number of people who were standing went up from 7 to $7 + 19$, that is, to 26.

Therefore there were now 26 people who had no seat.

3. The diagram shows an equilateral triangle, a square, and one diagonal of the square.

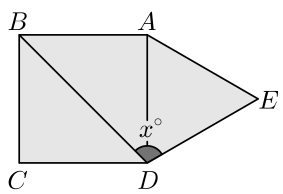
What is the value of x ?

A 105 B 110 C 115 D 120 E 135



Solution **A**

Let A, B, C and D be the vertices of the square, and A, D and E be the vertices of the equilateral triangle, as shown.



$\angle EDA = 60°$ because it is the angle of an equilateral triangle. $\angle ADC = 90°$ because it is the angle of a square. The diagonal DB bisects the angle ADC, and therefore $\angle ADB = 45°$.

It follows that $x° = \angle EDA + \angle ADB = 60° + 45° = 105°$.

Therefore

$$x = 105.$$

For investigation

3.1 The solution above takes it for granted that the diagonal DB bisects the angle ADC. Prove that this is correct.

4. The perimeter of the regular decagon P is 8 times the perimeter of the regular octagon Q.

Each edge of the regular octagon Q is 10 cm long.

How long is each edge of the regular decagon P?

A 8 cm B 10 cm C 40 cm D 60 cm E 64 cm

SOLUTION **E**

An octagon has eight edges. Therefore the perimeter of the regular octagon Q is $8 \times 10\text{ cm} = 80\text{ cm}$.

The perimeter of the regular decagon P is eight times that of the regular octagon Q and is therefore 8×80 cm, that is, 640 cm.

A decagon has 10 edges. Therefore the length of each edge of Q is given by

$$(640 \div 10)\text{ cm} = 64\text{ cm}.$$

FOR INVESTIGATION

4.1 The perimeter of the regular dodecagon R is 9 times that of the regular decagon P.

How long is each edge of the regular dodecagon R?

5. My train left Southampton at 06:15 and arrived in Birmingham at 08:48 later that morning.

How many minutes did the journey take?

A 153 B 193 C 233 D 1463 E 1501

SOLUTION **A**

It is 2 hours from 06:15 to 08:15. Because $48 - 15 = 33$, there are 33 minutes from 08:15 to 08:48. Hence, in total, the journey took 2 hours and 33 minutes.

There are 120 minutes in two hours. Therefore the journey took $120 + 33$ minutes, that is, 153 minutes.

FOR INVESTIGATION

5.1 Later the same day I left Birmingham by train at 19:04. My train arrived back at Southampton at 21:40. How many minutes did my return journey take?

5.2 On the same day it took me 15 minutes to walk from home to the station where I had a 5 minute wait for the 06:15 train. On my return it took me 15 minutes to walk from the station to my home.

For how long was I away from home?

6. The diagram shows a partially completed magic square, in which all rows, all columns and both main diagonals have the same total.

What is the value of $x + y$?

A 10 B 11 C 12 D 13 E 14

4		
	7	y
6	5	x

Solution **D**

Let w be the number in the top-right square, as shown in the figure.

4		w
	7	y
6	5	x

Because we are dealing with a magic square, the total of the numbers in the right-hand column is the same as the total of the numbers in the diagonal from the bottom-left square to the top-right square.

This gives

$$x + y + w = 6 + 7 + w.$$

Hence, by subtracting w from both sides, we deduce that

$$x + y = 6 + 7.$$

It follows that

$$x + y = 13.$$

For investigation

6.1 Note that in the above solution we found the value of $x + y$ without finding the value of either x or y separately.

Find the values of x and y. Hence find all the numbers in the magic square.

6.2 Complete the magic square shown on the right.

	3	13
9		

7. How many integers are greater than $20 + 18$ and also less than 20×18?

A 320 B 321 C 322 D 323 E 324

Solution **B**

We have $20 + 18 = 38$ and $20 \times 18 = 360$.

The integers that are greater than 38 and less than 360 are the integers from 39 to 359, inclusive. There are $(359 - 39) + 1$, that is, 321 of these integers.

For investigation

7.1 How many integers are there from 2018 to 9999, inclusive?

7.2 How many integers are there from -23 to 77, inclusive?

7.3 Explain why, in general, the number of integers from a to b, inclusive, where $a < b$, is $(b - a) + 1$.

8. Gill scored a goal half way through the second quarter of a 'teachers versus pupils' netball match.

At that point, what fraction of the whole match remained to be played?

A $\frac{1}{4}$ B $\frac{3}{8}$ C $\frac{1}{2}$ D $\frac{5}{8}$ E $\frac{3}{4}$

Solution **D**

Because $\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$, a half of a quarter is one eighth.

At the time when Gill scored the goal the first quarter and half the second quarter of the whole match had been completed. So the fraction of the game that had been completed was

$$\frac{1}{4} + \frac{1}{8} = \frac{2}{8} + \frac{1}{8} = \frac{3}{8}.$$

Therefore the fraction of the game that remained to be played was

$$1 - \frac{3}{8} = \frac{8}{8} - \frac{3}{8} = \frac{5}{8}.$$

For investigation

8.1 Gill scored a second goal one-third of the way through the third quarter of the netball match.

(a) At that point, what fraction of the whole match remained to be played?

(b) What fraction of the game elapsed between the first and second goals that Gill scored?

9. The approximate cost of restoring the Flying Scotsman was £4 million. This was about 500 times the cost of building the steam engine in 1923.

Roughly what did the engine cost to build?

A £800 B £2000 C £8000 D £20 000 E £80 000

SOLUTION **C**

There are 4 000 000 pounds in £4 million. Therefore, the cost of building the engine, in pounds, was roughly

$$\frac{1}{500} \times 4\,000\,000 = \frac{4\,000\,000}{500} = 8000.$$

So, roughly, the engine cost £8000 to build.

10. Adding four of the five fractions $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}, \frac{1}{9}$ and $\frac{1}{18}$ gives a total of 1.

Which of the fractions is not used?

A $\frac{1}{2}$ B $\frac{1}{3}$ C $\frac{1}{6}$ D $\frac{1}{9}$ E $\frac{1}{18}$

SOLUTION **C**

There are five different ways of selecting four of the five fractions given in the question. One method would be to try these different combinations in turn to find the four fractions that add up to 1.

However, it is much quicker to begin by adding all five of the fractions. This will give an answer greater than 1. We will then be able to see which fraction we need to remove to reduce the sum to 1.

By putting all of the fractions over the common denominator 18, we see that

$$\begin{aligned}\frac{1}{2}+\frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{18} &= \frac{9+6+3+2+1}{18} \\ &= \frac{21}{18} \\ &= \frac{7}{6} \\ &= 1+\frac{1}{6}.\end{aligned}$$

We see from this calculation that to obtain a total of 1 it is the fraction $\frac{1}{6}$ that is not used.

FOR INVESTIGATION

10.1 Find other examples of four different fractions of the form $\frac{1}{n}$, where n is a positive integer, whose total is 1.

11. How many digits are there in the correct answer to the calculation $123\,123\,123\,123 \div 123$?

A 4 B 6 C 8 D 10 E 12

Solution **D**

From the long division sum shown on the right, we see that

$$123\,123\,123\,123 \div 123 = 1\,001\,001\,001.$$

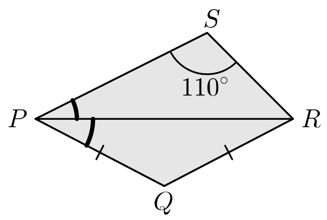
We see that there are 10 digits in this answer.

$$\begin{array}{r}
1\,001\,001\,001 \\
123\overline{)123\,123\,123\,123} \\
\underline{123} \\
0\,123 \\
\underline{123} \\
0\,123 \\
\underline{123} \\
0\,123 \\
\underline{123} \\
0
\end{array}$$

12. The diagram shows a quadrilateral $PQRS$ in which PQ and QR have the same length. Also PR bisects $\angle SPQ$, the ratio of $\angle SPR$ to $\angle PRS$ is $2:3$ and $\angle PSR = 110°$.

How large is angle PQR?

A 124° B 120° C 110° D 90° E 28°



Solution **A**

Because the angles of a triangle have sum 180° and $\angle PSR = 110°$, it follows that $\angle SPR + \angle PRS = 180° - 110° = 70°$.

The ratio of $\angle SPR$ to $\angle PRS$ is $2:3$. It follows that $\angle SPR = \frac{2}{5} \times 70° = 28°$.

Since PR bisects $\angle SPQ$, we deduce that $\angle RPQ = \angle SPR = 28°$.

Since $PQ = QR$, the triangle PQR is isosceles, and therefore $\angle PRQ = \angle RPQ = 28°$.

Thus the triangle PQR has two angles which are each 28°. Therefore, because sum of the angles of a triangle is 180°, we deduce that

$$\angle PQR = 180° - 28° - 28° = 124°.$$

For investigation

12.1 The solution above uses the following theorem: The angles at the base of an isosceles triangle are equal to one another.

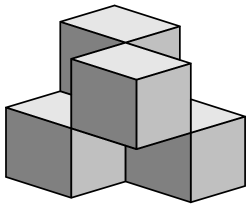
Find a proof of this theorem.

12.2 The theorem stated in Problem 12.1 is Proposition 5 of Euclid's *Elements*, Book 1. In the days when Euclid's *Elements* was used as a textbook in schools this proposition was known as the *pons asinorum*. Find out what this means, and why the theorem was given this name.

13. The diagram shows a shape made from four $3\,\text{cm} \times 3\,\text{cm} \times 3\,\text{cm}$ wooden cubes joined by their edges.

What, in cm^2, is the surface area of the shape?

A 162 B 180 C 198 D 216 E 234



Solution **D**

Method 1

The surface of the shape is made up of the full surface of each of the four cubes that make it up.

Each face of the $3\,\text{cm} \times 3\,\text{cm} \times 3\,\text{cm}$ cubes is a square with side length 3 cm. Hence the area of each of these faces is $3\,\text{cm} \times 3\,\text{cm} = 9\,\text{cm}^2$.

Each cube has 6 of these faces. Hence the surface area of each cube is $6 \times 9\,\text{cm}^2 = 54\,\text{cm}^2$.

Therefore the total surface area of the four cubes is $4 \times 54\,\text{cm}^2 = 216\,\text{cm}^2$.

Hence the surface area of the shape is $216\,\text{cm}^2$.

Method 2

If you look at the shape from above, or below, or from any of the four sides, you will see a 2×2 square.

That is, you will see four squares each with size $3\,\text{cm} \times 3\,\text{cm}$. So you see a total area of $4 \times 9\,\text{cm}^2$, that is $36\,\text{cm}^2$.

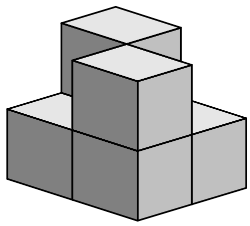
Because this is the case for each of the six directions (top, bottom and four sides) the total surface area of the shape is given by

$$6 \times 36\,\text{cm}^2 = 216\,\text{cm}^2.$$

For investigation

13.1 The shape shown uses six $1\,\text{cm} \times 1\,\text{cm} \times 1\,\text{cm}$ wooden cubes. It has been made by removing two such cubes from the top layer of a $2\,\text{cm} \times 2\,\text{cm} \times 2\,\text{cm}$ cube.

What, in cm^2, is the surface area of the shape?



13.2 What is the smallest possible surface area of a shape made from four $3\,\text{cm} \times 3\,\text{cm} \times 3\,\text{cm}$ cubes?

13.3 What is the smallest possible surface area of a shape made from six $1\,\text{cm} \times 1\,\text{cm} \times 1\,\text{cm}$ cubes?

14. Billy has three times as many llamas as lambs. Milly has twice as many lambs as llamas.

They have 17 animals in total.

How many of the animals are llamas?

A 5 B 6 C 7 D 8 E 9

Solution **E**

Suppose that Billy has b lambs and hence $3b$ llamas. Then, in total, Billy has $4b$ animals.

Suppose that Milly has m llamas and hence $2m$ lambs. Then, in total, Milly has $3m$ animals.

Because the total number of animals is 17, we have

$$4b + 3m = 17.$$

We have two unknowns but just one equation. This equation has infinitely many different solutions. However, in this problem b and m are numbers of animals, and so each of them is a non-negative integer. This enables us to find their values.

Here b and m are non-negative integers. It follows that $4b \leq 17$ and hence $b \leq 4$. So the only possible values for b are 0, 1, 2, 3 and 4.

It is easy to check that, of these values of b, only when $b = 2$ is the corresponding value of m an integer.

When $b = 2$, we have

$$4 \times 2 + 3m = 17,$$

that is,

$$8 + 3m = 17,$$

and hence

$$3m = 9,$$

from which it follows that

$$m = 3.$$

Since $b = 2$, Billy has $3 \times 2 = 6$ llamas. Since $m = 3$, Milly has 3 llamas.

Therefore $6 + 3 = 9$ of the animals are llamas.

For investigation

14.1 Check that for $b = 0, 1, 3, 4$, the value of m that satisfies the equation

$$4b + 3m = 17,$$

is not an integer.

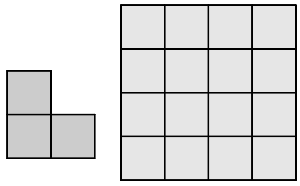
14.2 a and b are positive integers and $7a + 5b = 49$. Find the values of a and b.

14.3 a and b are positive integers and $23a + 17b = 320$. Find the values of a and b.

15. Beatrix places copies of the L-shape shown on a 4×4 board so that each L-shape covers exactly three cells of the board. She is allowed to turn around or turn over an L-shape.

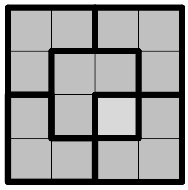
What is the largest number of L-shapes she can place on the board without overlaps?

A 2 B 3 C 4 D 5 E 6



Solution **D**

The figure shows one way of placing 5 L-shapes on the board without overlaps.



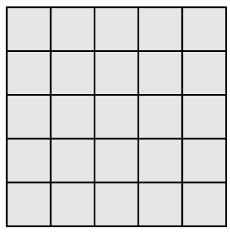
The 4×4 board is made up of 16 cells. The L-shape is made up of 3 cells.

Therefore, however 5 L-shapes are placed on the board without overlapping, there will be just one cell that remains uncovered. So there will not be space for one more L-shape.

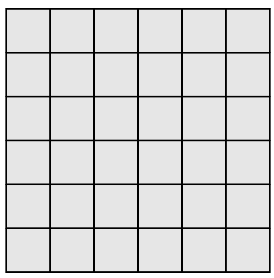
Hence, the largest number of L-shapes that Beatrix can place on the board without overlaps is 5.

For investigation

15.1 What is the largest number of L-shapes that Beatrix can place on a 5×5 board without overlaps?



15.2 What is the largest number of L-shapes that Beatrix can place on a 6×6 board without overlaps?



15.3 What can you say in general about the number of L-shapes Beatrix can place on an $n \times n$ board without overlaps?

16. How many pairs of digits (p, q) are there so that the five-digit integer '$p869q$' is a multiple of 15?

A 2 B 3 C 4 D 5 E 6

Solution **E**

We have $15 = 3 \times 5$. Also, 3 and 5 have no common factors other than 1. It follows that the multiples of 15 are just the numbers that are multiples of both 3 and of 5.

For '$p869q$' to be a multiple of 5 we require that its units digit q is either 0 or 5.

For '$p869q$' to be a multiple of 3 we require that the sum of its digits is a multiple of 3.

We first consider the case where q is 0. In this case the sum of the digits of '$p869q$' is given by

$$p + 8 + 6 + 9 + 0 = p + 23.$$

Because '$p869q$' is a five-digit integer, p is not 0. So the possible values of p are 1, 2, 3, 4, 5, 6, 7, 8 and 9. It is easily checked that there are three of these digits, namely 1, 4 and 7, for which $p + 23$ is a multiple of 3.

Similarly, when q is 5, the sum of the digits of '$p869q$' is

$$p + 8 + 6 + 9 + 5 = p + 28.$$

In this case there are also three choices of p, namely 2, 5 and 8, for which $p + 28$ is a multiple of 3.

It follows that the number of pairs of the digits (p, q) for which '$p869q$' is a multiple of 15 is $3 + 3$, that is, 6.

[Note: as we have found above, these pairs are $(1, 0)$, $(4, 0)$, $(7, 0)$, $(2, 5)$, $(5, 5)$ and $(8, 5)$.]

For investigation

16.1 How many pairs of digits (p, q) are there so that the six-digit number '$p6789q$' is a multiple of 15?

16.2 How many pairs of digits (p, q) are there so that the five-digit number '$p547q$' is a multiple of 165?

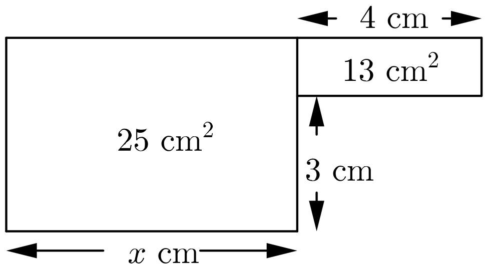
16.3 The solution above uses the fact that the criterion for whether a positive integer is a multiple of 3 is that the sum of its digits is a multiple of 3.

Find out why this criterion is correct (either prove that the criterion is correct, or find an explanation in a book or on the web, or ask your teacher).

17. The areas of the two rectangles in the diagram are $25\,\text{cm}^2$ and $13\,\text{cm}^2$ as indicated.

What is the value of x?

A 3 B 4 C 5 D 6 E 7



Solution **B**

The rectangle with area $13\,\text{cm}^2$ has width $4\,\text{cm}$. Therefore the height of this rectangle is $(13 \div 4)\,\text{cm}$, that is, $\dfrac{13}{4}$ cm.

It follows that the height of the rectangle with area $25\,\text{cm}^2$ is $\left(3 + \dfrac{13}{4}\right)$ cm.

Because this rectangle has area $25\,\text{cm}^2$, its width is $\left(25 \div \left(3 + \dfrac{13}{4}\right)\right)$ cm.

Now $3 + \dfrac{13}{4} = \dfrac{12}{4} + \dfrac{13}{4} = \dfrac{25}{4}$. It follows that $25 \div \left(3 + \dfrac{13}{4}\right) = 25 \div \dfrac{25}{4} = 25 \times \dfrac{4}{25} = 4$.

Hence the value of x is 4.

18. Between them, the two five-digit integers M and N contain all ten digits from 0 to 9.

What is the least possible difference between M and N?

A 123 B 247 C 427 D 472 E 742

Solution **B**

The difference between M and N is either $M - N$ or $N - M$, according as $M > N$ or $N > M$.

We can assume that $M > N$, since otherwise we can swap the roles of M and N in the following argument.

To obtain the least possible value of the difference $M - N$ we want the difference between the first digit of M and the first digit of N to be as small as possible. Therefore we seek a solution in which the first digit of M is just 1 more than the first digit of N. We then arrange the remaining digits to make M as small as possible, and N as large as possible.

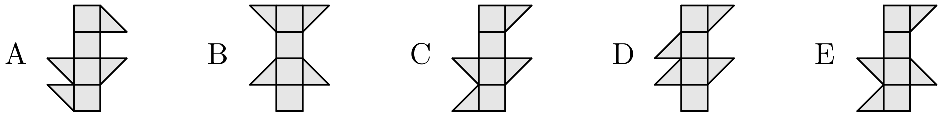
Let the first digit of M be d. Provided that d is none of 0,1,2,3 and 4, the smallest possible value of M is 'd0123'.

Similarly, if the first digit of N is e and e is none of 6,7,8 and 9, the largest possible value of N is 'e9876'.

Conveniently, these choices for M and N enable us to take d to be 5 and e to be 4.

This gives $M = 50123$ and $N = 49876$. It follows that the least possible difference between M and N is $50123 - 49876 = 247$.

19. Which one of these could be folded to make a cube?

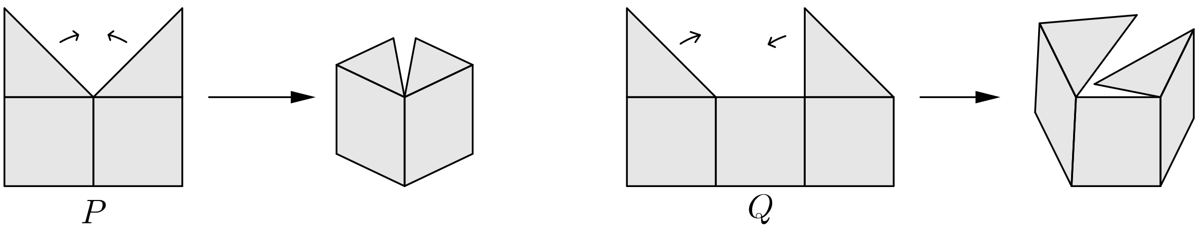


Solution **C**

Each of the shapes given in the question consists of a strip of four squares with, on either side, two triangles which together might make up a square. For convenience we refer to these as *flaps*.

The four squares will fold to make four faces of a cube. The shape can be folded to make an entire cube provided that each pair of flaps will fold together to make a square face.

It can be seen that flaps in either of the configurations P or Q, as shown in the figure below, will fold to make a square face.



The shape of option C has a pair of flaps in configuration P on the left-hand side, and a pair of flaps in configuration Q on the right-hand side. It follows that this shape could be folded to make a cube.

In the context of the JMC we can stop here. However, for a complete answer we would need to show why none of the other shapes could be folded to make a cube.

We leave it to the reader to check this for themselves.

For investigation

19.1 Show that each of the shapes A, B, D and E cannot be folded to make a cube.

20. A drawer contains ten identical yellow socks, eight identical blue socks and four identical pink socks.

Amrita picks socks from the drawer without looking.

What is the smallest number of socks she must pick to be sure that she has at least two pairs of matching socks?

A 5 B 6 C 8 D 11 E 13

SOLUTION **B**

We first note that 5 socks is not enough. For example, if Amrita picks 3 yellow socks, 1 blue sock and 1 pink sock, she has just one pair of matching yellow socks, and no other pairs.

Now suppose Amrita picks 6 socks. We show that in this case she must have at least two pairs of matching socks.

If Amrita has picked at least 4 socks of the same colour, she will have two pairs of socks of that colour.

Suppose she has picked just 3 socks of one particular colour, say 3 yellow socks. Then she has a pair of yellow socks, an odd yellow sock, and three other socks each of which is either blue or pink. It follows that she has either at least two blue socks or at least two pink socks, and hence she has a pair of socks of a second colour. So in this case she has two pairs of matching socks,

The remaining case is when Amrita has not picked as many as three socks of any one colour. This means that she has at most two socks of each of the colours. Since she has six socks the only possibility is that she has two socks of each of the three colours. So in this case she has three pairs of matching socks.

Therefore we have shown that 5 socks is too few, but picking 6 socks guarantees than Amrita has at least two pairs of matching socks.

FOR INVESTIGATION

20.1 What is the smallest number of socks Amrita needs to pick to be sure that she has at least one pair of matching socks?

20.2 What is the smallest number of socks Amrita needs to pick to be sure that she has at least three pairs of matching socks?

20.3 Suppose that in addition to the socks mentioned in the question Amrita's drawer also contains twelve identical green socks.

What is the smallest number of socks Amrita needs to pick to be sure that she has at least two pairs of matching socks?

21.

There are ——— vowels in this short sentence.

Which of the following options should replace "———" to make the sentence in the box true?

A twelve B thirteen C fourteen D fifteen E sixteen

SOLUTION **D**

We note that initially there are 12 vowels in the box.

In the following table we show for each of the words that we might use to replace the gap, the number of vowels added, and the total number of vowels in the completed sentence.

initial number of vowels	word used	number of vowels added	total number of vowels
12	twelve	2	14
12	thirteen	3	15
12	fourteen	4	16
12	fifteen	3	15
12	sixteen	3	15

We see that *fifteen* is the only one of the given options that makes the sentence true when it replaces the gap.

FOR INVESTIGATION

21.1 Find a number written in words (for example, *eleven* or *twenty-three*) that could be used to replace "———" in the following sentence to make it true.

There are ——— consonants in this sentence.

Note: the answer will depend on whether you regard the letter y as a consonant or a vowel. There is at least one correct answer in either case.

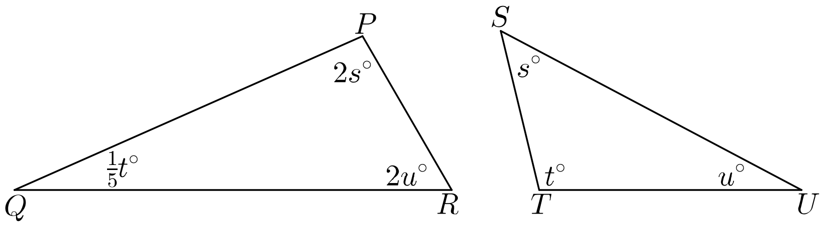
22. In the triangles PQR and STU, $\angle RPQ = 2 \times \angle UST$, $\angle PRQ = 2 \times \angle SUT$ and $\angle RQP = \dfrac{1}{5} \times \angle UTS$.

How large is $\angle UTS$?

A $90°$ B $100°$ C $120°$ D $150°$
E more information needed

SOLUTION **B**

Let the angles of the triangle STU be $s°$, $t°$ and $u°$ as shown in the figure.



It follows that the angles of the triangle PQR are $2s°$, $\frac{1}{5}t°$ and $2u°$, as shown.

The angles of a triangle have sum $180°$. Therefore

$$s + t + u = 180 \qquad (1)$$

and

$$2s + \tfrac{1}{5}t + 2u = 180. \qquad (2)$$

By multiplying equation (1) by 2 we obtain

$$2s + 2t + 2u = 360. \qquad (3)$$

Then, by subtracting equation (2) from equation (3), we deduce that

$$\left(2 - \tfrac{1}{5}\right)t = 180,$$

that is,

$$\tfrac{9}{5}t = 180.$$

Therefore

$$\begin{aligned} t &= \tfrac{5}{9} \times 180 \\ &= 100. \end{aligned}$$

It follows that $\angle UTS = 100°$.

Note that we can deduce that $s + u = 80$, but it is not possible to deduce the individual values of s and u.

FOR INVESTIGATION

22.1 Suppose that $\angle RPQ = 3 \times \angle UST$, $\angle PRQ = 3 \times \angle SUT$, and $\angle RQP = \dfrac{1}{2} \times \angle UTS$.

Find $\angle UTS$ in this case.

23. Ali wants to fill the empty squares so that the number in each square after the fourth from the left is the sum of the numbers in the four squares to its left.

2		0		1		8	

What number should Ali write in the final square?

A 16 B 8 C 4 D 2 E 1

Solution **A**

Let the numbers that Ali needs to place in the empty squares be p, q, r and s, as shown in the figure.

2	p	0	q	1	r	8	s

Because each of the numbers 8 and s is the sum of the four numbers that precede it,

$$0 + q + r + 1 = 8 \qquad (1)$$

and

$$q + 1 + r + 8 = s. \qquad (2)$$

By subtracting equation (1) from equation (2), we have

$$(q + 1 + r + 8) - (0 + q + r + 1) = (s - 8),$$

and hence,

$$8 = s - 8.$$

Therefore, by adding 8 to both sides, we deduce that

$$16 = s.$$

It follows that Ali should write 16 in the final square.

For investigation

23.1 Determine the values of p, q and r, as defined in the above solution.

23.2 Show that it is possible to put numbers in the empty squares so that the number in each square after the fourth from the left is the sum of the numbers in the four squares to its left.

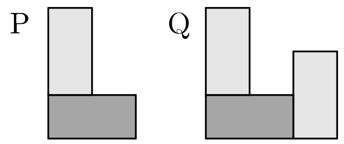
1		1		1		1	

23.3 Show that for each choice of values for a, b, c and d, it is possible to put numbers in the empty squares so that the number in each square after the fourth from the left is the sum of the numbers in the four squares to its left.

a		b		c		d	

24. The shapes P and Q are formed from two and three identical rectangles, respectively. Their perimeters are 58 cm and 85 cm respectively.

What is the perimeter of one of the rectangles?



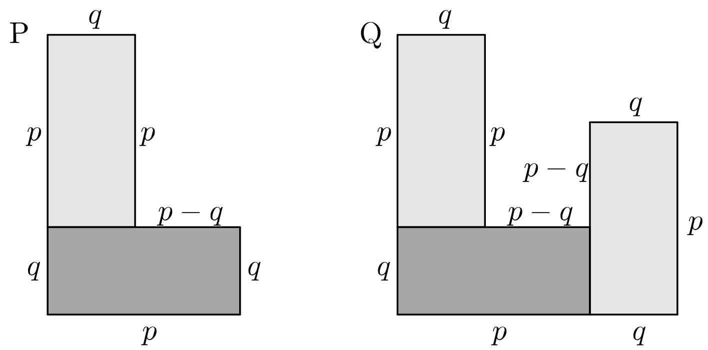
A 30 cm B 31 cm C 32 cm D 33 cm E 34 cm

SOLUTION **B**

Let the longer sides of each of the identical rectangles have length p cm and the shorter sides have length q cm.

Note that the perimeter of each of the rectangles is $(2p + 2q)$ cm.

In the figure below we have indicated the lengths, in centimetres, of the line segments that make up the boundaries of the shapes P and Q.



From this figure we see that the perimeter of P is given by

$$p + q + p + q + (p - q) + p + q = 4p + 2q,$$

and that the perimeter of Q is given by

$$p + q + p + q + p + q + (p - q) + (p - q) + p + q = 6p + 2q.$$

(Note that in each case we have started at the top left of the figure and have added up the lengths going round the figure anticlockwise.)

We now use the information given in the question to deduce that

$$4p + 2q = 58,$$

and

$$6p + 2q = 85.$$

By subtracting the first of these equations from the second, we obtain

$$2p = 27.$$

It follows that

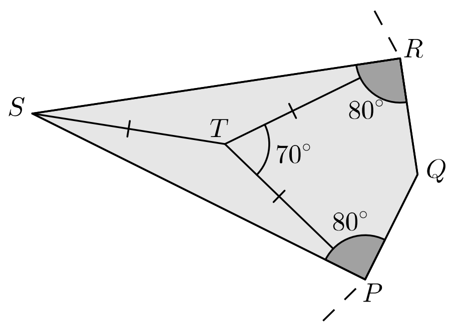
$$\begin{aligned} 2p + 2q &= (4p + 2q) - 2p \\ &= 58 - 27 \\ &= 31. \end{aligned}$$

Hence the perimeter of one of the rectangles is 31 cm.

25. In the diagram PQ and QR are sides of a regular n-sided polygon, $\angle SPQ = \angle SRQ = 80°$, $\angle PTR = 70°$ and $PT = ST = RT$.

What is the value of n?

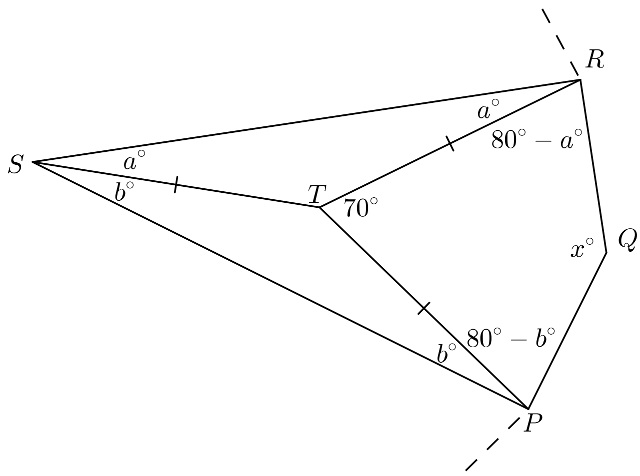
A 15 B 18 C 20 D 24 E 30



SOLUTION **D**

The exterior angle of a regular n-sided polygon is $\left(\frac{360}{n}\right)^\circ$. [Can you prove this?] Our strategy will be to find the interior angle, $\angle PQR$, of the regular polygon. Then we use this to calculate the exterior angle of the polygon and hence the corresponding value of n.

We let $\angle PQR$ be $x°$, $\angle RST$ be $a°$ and $\angle PST$ be $b°$. [Because of the symmetry of the figure $a = b$, but it turns out that we don't need to use this fact.]



Because $ST = RT$, the triangle RST is isosceles. Hence $\angle TRS = \angle RST = a°$.

Because the angles in a triangle have sum 180°, we see from the triangle RST that $\angle STR = 180° - 2a°$.

Similarly, $\angle STP = 180° - 2b°$.

The sum of the angles at the point T is 360° and therefore $70° + (180° - 2a°) + (180° - 2b°) = 360°$. It follows that $2a° + 2b° = 70°$, and hence that $a° + b° = 35°$.

We now use the fact that the sum of the angles in the quadrilateral $PQRT$ is 360°.

Because $\angle SRQ = 80°$ and $\angle SRT = a°$, we have $\angle QRT = 80° - a°$. Similarly $\angle QPT = 80° - b°$. Therefore

$$x° + (80° - a°) + (80° - b°) + 70° = 360°.$$

It follows that

$$x° = 360° - 80° - 80° - 70° + a° + b° = 130° + a° + b° = 130° + 35° = 165°.$$

Because the interior angle of the polygon is 165°, its exterior angle is $180° - 165° = 15°$. It follows that

$$\frac{360°}{n} = 15°$$

and therefore

$$n = \frac{360}{15} = 24.$$